Structural Approach in Teaching as foundation for Sustainable Arithmetic learning – SATSA

Research plan

The planned project SATSA stems from a growing body of research emphasizing the importance of early interventions as foundational for prosperous development of number knowledge and arithmetic skills (Duncan et al. 2007). Early interventions often show a positive success rate (Clements & Sarama 2011; Gersten, Jordan & Flojo 2005), but fade-out effects are also very common (Wang et al. 2016). Critical for educational science is thereby how to design and implement interventions that result in a conceptually solid understanding of numbers that leads to sustainable successful executive skills in arithmetic problem solving. The key to conceptualizing powerful interventions, we suggest, is to direct attention to *the quality of teaching* number concepts in the early years, that is, how conceptual growth is facilitated in the teaching that is offered to children. To find the key features of successful ways to teach that result in knowledge and skills that are *developable* and thus *sustainable* in a prolonged perspective is a highly desirable goal. This question is also highlighted in the contemporary debate in Sweden concerning the first mandatory year of education (preschool class), which organization, learning goals and ways of teaching are under review, commissioned by the Swedish Government (Dir. 2020:24). Our planned study focusing quality of teaching in early years education, is a contribution to the debate, offering scientific insights from empirical theory-based research.

In consensus with a large body of research we assume that successful arithmetic competence relies on the ability to handle numbers as structured in a part-whole-relationship (Baroody & Purpura 2017; Fuson 1992; Venkat, Askew, Watson & Mason 2019). In a recent project we implemented a *structural approach* to teaching arithmetic to 5-yearolds. This project was successful in terms of the children learning to use number relations as an outset for solving simple arithmetic tasks. Many learnt the relation between numbers (Björklund, Ekdahl & Runesson Kempe, in review), thus, having developed a conceptual understanding of number concepts. However, in a follow up interview one year after the intervention, when the children had undertaken one year of mathematics education in preschool class, as 6-yearolds, many had abandoned their conceptually powerful strategies to solve arithmetic tasks and instead made use of more primitive procedural counting strategies, which are not developable in a long term perspective (Ellemor-Collins & Wright 2009; Schollar 2015). Two critical issues then arose: what did the teaching in the year following the successful intervention afford (or not afford) for altering the children's ways of understanding and using numbers, and how could the structural approach be improved to be sustainable over a prolonged period of time? These issues guide our here proposed project plan.

Purpose and aims

The aim of the planned project is to characterize what makes mathematics teaching concerning number concepts and basic arithmetic skills among 6-yearolds *sustainable and developable*. The specific research questions to be answered are:

RQ1: What foundation for further learning arithmetic is provided in preschool class mathematics education?
RQ2: To what extent is it possible to enhance pupils' number knowledge and arithmetic skills as to be sustainable, through teaching with a structural approach?
RQ3: What are critical features for prosperous arithmetic skills development in the structural approach in teaching?

In the planned project we will conduct observations of teaching practices in Swedish preschool classes, we will implement a structural approach in teaching number concepts and arithmetic strategies and we will assess pupils' developing number knowledge, to answer the aim and research questions. RQ1 is answered by observing preschool class education, to gain a large-scale overview of the teaching quality in Swedish mathematics education with 6-yearolds. From these observations we identify classes who will be offered participation in an intervention program aiming to enhance their teaching by implementing theory-driven structural approaches. RQ2 will be answered by following the arithmetical development among the pupils participating in the intervention program (intervention groups), compared to pupils involved in their ordinary mathematics education (comparison group). The intervention offers two distinct programs for implementing a structural approach (see Project description). RQ3 will then be answered through a thorough analysis of the teaching enacted in accordance with the two programs, both of them implementing a structural approach but based in different ideas of how to facilitate pupils learning to see number relations. This will give us a deeper insight to the core features of a structural approach in teaching and empirically reveal to what extent and on what groundings the approach facilitates successful arithmetic development.

The object of our research is teach*ing* (not teach*ers*) and how teaching is reflected in learning outcomes. That is, our interest in researching teaching goes beyond frequently used categories of teaching practice such as teacher-centered vs student-centered, inquiry-based vs direct instruction or teachers' characteristics (beliefs, subject knowledge, years of experiences etc.), and instead focus on the classroom interactions around a specific content directed towards facilitating students' achievement of learning goals (see Hiebert & Grouws, 2007).

State-of-the art

Mathematics teaching is a large field of knowledge, why we here focus on the theoretical principles for the structural approach in teaching arithmetic, how research has conceptualized this approach and empirical findings of its efficacy for children's arithmetic development.

The notion of structure in mathematics is by Venkat, Askew, Watson and Mason (2019) described as the mathematical relations between elements. Emergent structure is observed among younger children as "…an arrangement of elements – symbols or images – in some particular organization that serves to stress a mathematically appropriate relationship" (p. 14). (True) mathematical structure, in contrast, involves more general relations, going beyond single cases, and are linked to each other (generic), including general properties that are applicable across classes of examples.

Structural awareness is considered a key aspect of mathematical understanding and has been found to be a predictor of later arithmetic competences (Mulligan & Vergnaud 2006; Papic, Mulligan & Mitchelmore; 2011; Venkat, et al. 2019). Seeing structure allows for making use of relations between quantities, partitioning of numbers, relations between operations, as well as multiplicative reasoning (Carraher, Schliemann, Brizuela & Ernst 2006; Mulligan, Mitchelmore, English & Crevensten 2013, Warren 2003). Children who show a higher structural awareness when encountering general mathematical content are also found to more easily learn the properties of numbers (Mulligan & Mitchelmore 2009; Papic et al. 2011; Mulligan et al. 2013).

Teaching with a structural approach means attending to developing children's awareness of similarities and differences, and ability to create units and to reason about parts and whole simultaneously (Davydov 1982; Schmittau 2003, Zhou & Peverly 2005). Consequently, emphasis lies on composition and decomposition of numbers as the outset for solving addition and subtraction tasks, rather than operating on single units. Systematic organization of (sets of) items in examples involving actions that draw the learners' attention to adequate relations and properties of numbers, further allows the learners to see local relations (Venkat et al. 2019), which is assumed to develop their mathematical thinking (Mason, Stephens & Watson 2009). Intervention programs taking a structural approach to teaching numbers in early grades (Venkat et al. 2019) show a great proportion of pupils moving beyond counting-in-one strategies and learning to use number relations and structure in solving arithmetic tasks. Learning outcomes further show pupils succeeding to a greater extent than control groups in solving complex problems, e.g. missing start/missing addend problems (Polotskaia & Savard 2018) and demonstrating a powerful understanding of the *mathematical structure* of arithmetic problems (Sensevy et al. 2018). The structural approach is thus theoretically driven, but has in empirical studies shown positive outcomes for young pupils' learning, particularly concerning their conceptual knowledge of numbers.

Research (Cheng 2012) further shows that those strategies children learn before formal primary mathematics education (as 5-7 year-olds) are essential, since they form the basis for how the child approaches a larger number range and more complex tasks and early learnt strategies are found to be hard to abandon. This calls for an inquiry of how conceptually powerful ways of teaching number concepts and arithmetic are conducted when children are in their midst of learning the basic ideas of numbers as relational and how to use strategies as to be conceptually prosperous and developable.

Significance and scientific novelty

The project bears significance in advancing the knowledge of how to facilitate pupils' learning. The structural approach builds on theories of children's concept development, which we take further by relating to Vygotskian traditions (Davydov 1982; Schmittau 2003) and studies of pupils' ways of experiencing numbers as an outset for teaching rather than teaching counting strategies that risk becoming procedural (Neuman 1987). Research within the field of arithmetic learning has to a large extent built on the idea that counting strategies and learning trajectories observed in children are natural paths to excellence (e.g. Baroody 1987; Fuson 1992). The structural approach is a reaction to this, emphasizing a conceptual understanding to be foundational for learning to use powerful strategies, but less is known *how* this knowledge develops through the teaching. Following this, new ways of designing teaching that facilitates conceptual growth in children rather than teaching them to merely use more and more advanced strategies are necessary to implement but also to investigate. Thus, the project SATSA offers an alternative to the dominating counting-based teaching approaches and will put the structural approach to the test in empirical study to scrutinize the theoretical foundation and its benefits and limitations.

Concept development and learning of arithmetic skills are not isolated from context, it is influenced by what the teaching is offering the child to discern. Our contribution to the field of research and current debate in education is the focus on the *teaching,* and the *mathematical content* that is afforded to the pupils participating in the educational act. This focus is a neglected issue in didactic discussions (Cai, Morris, Hohensee, Hwang, Robison & Hiebert, 2017). For instance, considering the special form of education that 6-yearolds take part of in the Nordic countries, governments have made it a mandatory task that every child is to be screened for mathematical competence in the beginning and at the end of the school year, to identify children at risk and the teacher gain an overview of children's knowledge and skills[1]. However, this kind of screening is attending to children's knowledge and even though they are based in research they focus on the child, not the learning opportunities afforded to the child. Thus, if children lack in some knowledge area, the teacher will be able to identify the deficit, but the teacher will not know if it is the teaching that is not affording the child sufficient opportunities to learn. Our research will thereby add to the systematic work of enhancing quality in education by addressing a critical aspect (teaching) not sufficiently covered before. Due to the practice-based approach in our research design we furthermore see results immediately implemented in practice, for the benefit of the participating pupils and teachers and the field of education, which will be provided with empirically proven and theory-driven knowledge of teaching, instantly. This is one of the strengths of our research, that theory and practice are challenged and developed simultaneously. In a long-term perspective we anticipate enhanced quality in mathematics teaching in the first year of compulsory education in Sweden.

There are few mathematics intervention studies conducted in the Swedish preschool class context, particularly large-scale studies are scarce. Interventions with research purposes do however show promising results as in the pupils learning to solve arithmetic tasks, suggesting that 6-yearolds are in a critical phase of their mathematical development. One of these

[1] "Hitta matematiken" is a mandatory screening instrument for Swedish preschool class, covering number concept, patterns, problem-solving and spatial abilities (Skolverket, 2019). "Lukimat" is an instrument for screening mathematical competence in Finland, preschool class (Niilo Mäki Institute). The instrument is not mandatory to use but teachers are obliged by law to identify children in need of extended support. "Kartleggingsprøve i rekning" is the Norwegian screening instrument to be used by teachers to identify children at risk (Utdanningsdirektoratet).

programs (Sterner, Wolff & Helenius 2019) is based on ideas such as emphasizing numbers through reasoning in collaborative group work and using multiple representations for e.g. patterns and number line. Our project adds to these kind of studies with detailed insights of *how* such representations can become facilitative resources, that is, how number relations may come through in theory-driven systematic use of artefacts and teaching actions.

The research interest in the structural approach is shared with other research groups, focusing on the teaching and learning of basic arithmetic. We have a joint intention to deepen the knowledge of structural approach in mathematics education with the research group at WITS, University of the Witwatersrand, South Africa, lead by prof. Hamsa Venkat. The planned project in Sweden is a parallel project to planned projects to be conducted in South Africa. The contexts, policies and prerequisites for teaching about numbers and arithmetic are very different in these projects. But deeper knowledge of teaching with a structural approach, considering its core features and empirically proven principles, will give a stronger foundation for further joint projects that bridges over contextually influencing aspects.

Preliminary and previous results

A recent research project (FASETT supported by the Swedish Research Council under Grant no. 721-2014-1791) implemented a structural approach to teaching number concepts and arithmetic skills in preschool with 5-yearolds. The project showed positive effects in children's competence to solve arithmetic tasks that were consistent over a prolonged period of time (Kullberg, Björklund, Brkovic & Runesson Kempe, 2020). Nevertheless, a qualitative follow-up of the preschool study showed that even though the children during the intervention learned to attend to the number structure of arithmetic tasks, many turned to counting-based strategies one year later after participating in regular preschool class education (Björklund, in review). Thus, we have evidence from earlier research that teaching in accordance with a structural approach is prosperous for the development of conceptually powerful knowledge of numbers, but our findings also show that these effects are not necessarily stabile.

Results from Ekdahl (2019) reveal that when taking a structural approach to teaching it matters which representations are offered and if examples are provided with *systematicity*. Also, it is shown the importance of making *connections* (verbally and gesturally) to emphasize number relations and key features associated with a structural approach. Pupils in classes where the teacher used more connecting actions to direct learners' attention to connections between examples and systematicity were found to perform better compared to other classes (Ekdahl, Venkat & Runesson 2016). These are empirical findings of successful ways of enacting a structural approach, but need to be replicated and scaled up to be validated.

A thorough investigation of the relation between what is taught through a structural approach and what children actually learn in terms of more advanced ways of experiencing number meaning (Björklund et al. in review) shows potential for using the theoretical principles of variation theory of learning as a framework for analyzing *teaching affordances.* Furthermore, our way of implementing professional development through collaborative practice-oriented research has in earlier studies conducted in similar ways as the planned SATSA (FASETT funded by Swedish Research Council grant no. 721-2014-1791, DUTTA funded by Swedish Institute for Educational Research, grant no. 2018-00014, and EXTENT funded by Swedish Institute for Educational Research, grant no. 2018-00038) shown to be successful, in terms of teachers developing their ability to reflect on their teaching through the iterative processes that facilitates their making the theoretical principles their own (Björklund & Ekdahl, in review).

Project description

Theory. The core notion in our project is teaching. Teaching is commonly defined as an activity where one person intends to facilitate learning by another (Gage 1978). However, teaching is never one-directional, it is rather relational in that the teacher and learner participate in an act of discernment directed towards a shared content. How the learner experiences that content is to be widened and deepened through the teaching act, in which the teacher provides experiences that liberates new ways of experiencing the same content. In this way, we consider learning to be a result of changes in the learner's way of experiencing a certain content (Marton & Booth 1997). Thus, it becomes important to understand how the teaching act affords necessary experiences to be made by the learner. Hiebert and Grouws (2007) claim it being a difficult task to link teaching with learning, since most studies fail in measuring and analysing the same object in teaching and in learning. However, variation theory of learning (see below) contributes a model of description of the teaching-learning relation, where the key is the learner's way of experiencing what is to be learnt.

According to variation theory, how something is experienced depends on the aspects that come to the fore of attention and are discerned (Marton, 2015; Marton & Booth, 1997). When an aspect not previously discerned is presented as a pattern of variation, his/her ways of experiencing this object of learning may be changed, thereby perceiving the world in a more differentiated way (e.g. Gibson & Gibson, 1955). These principles – discernment and variation – can be used to analyze what is afforded to learn in a given teaching activity as a means to find out what is made possible for children to experience and learn. We suggest that these theoretical notions offer a coherent foundation that is possible to apply to data in order to relate what is taught and learnt. Through a very close analysis of the enacted object of learning in terms of how necessary aspects are made possible to discern by means of variation, together with analyzing learners' ways of experiencing (described in terms of discerned aspects), we can study how our intervention is reflected in the development of pupils' understanding of numbers. Thus, in what ways the structural approach affords number concepts to be learnt.

Methods. The project is designed as an intervention study including: Part I: observations of preschool class teaching, and Part II: assessment of children's number knowledge, and implementation of structural approach in teaching number concepts and arithmetic strategies (see Figure 1). Methods for sample selection is included in the description of the project design.

<table>
<tr><td rowspan="2">Part I</td><td colspan="3">Observations of teaching practices, n = 100 classes</td></tr>
<tr><td colspan="3">Selection for intervention program participation</td></tr>
<tr><td rowspan="5">Part II</td><td colspan="3">Pre-assessment of children's number knowledge, n ≈ 900 children from 45 classes</td></tr>
<tr><td>Implementation of intervention program A, n=15 classes</td><td>Implementation of intervention program B, n=15 classes</td><td>Comparison group, business as usual, n=15 classes</td></tr>
<tr><td>Observations of teaching practices</td><td>Observations of teaching practices</td><td>Observations of teaching practices</td></tr>
<tr><td colspan="3">Post-assessment of children's number knowledge</td></tr>
<tr><td colspan="3">Delayed assessment of children's number knowledge</td></tr>
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Figure 1. Overview of the project's empirical phases and participants.

Observations of preschool class teaching

Observations of the teaching practices here means to focus on a) tasks and choice of examples, b) artefacts, c) representations, and d) talk and gestures, as four key strands for the quality of enacted teaching, with explicit focus on what is made object of learning and how this object of learning is afforded to the pupils in the teaching. The observation protocol we use is developed by Venkat and Askew (2018) and fits our purposes well as it is directed towards how the teaching enables pupils to become aware of necessary aspects of numbers and arithmetic (Askew et al. 2019). The protocol consists of four quality levels that are applied to the key strands mentioned above (see example of the strand 'artefacts' in Figure 2).

Mediating artefacts			
No artefacts used or artefacts that are problematic / inappropriate	Unstructured artefacts used in unstructured ways	Structured artefacts used in unstructured ways	Structured artefacts used in structured ways / unstructured artefacts used in structured ways
0	1	2	3

Figure 2. Example of key component 'artefacts' in the protocol, by Venkat and Askew (2018).

Observations with the protocol will be done in 100 classes in the southern part of Sweden, selected to control for a variation in socio-economic factors. Observations are done by researchers and research assistants (trained students) doing on-site visits. The observations provide data for analyzing the teaching quality in both quantitative and qualitative manners. From the sample of 100 classes we offer 30 of those classes scoring the lowest mean score on the observation protocol to participate in the designed intervention programs. One group of 15 classes is randomly selected as comparison group. We will seek balance in socio-economic conditions and gender among the pupils. Observations will also be done during the intervention to validate the progress of implementing the programs.

Assessment of children's number knowledge

Since teaching quality is related to the learning outcome of those who participate in the teaching activities, we assess pupils' development of number knowledge in the intervention and comparison classes (with their legal guardians' consent). The pupils are individually assessed at three occasions: before, right after and delayed one year after the intervention. The teachers administrate the assessments of the pupils in accordance with methods that encourage reasoning rather than producing rapid correct answers, which have been tested and used in earlier studies (Kullberg et al. 2020). The purpose is to identify children's pre-existing ways of experiencing numbers and how their ways of experiencing numbers change during the intervention, and to what extent any prosperous development is sustained or further developed in a prolonged time period. This will be analyzed in accordance with variation theory principles which has earlier been found to be proficient for similar purposes (Björklund & Runesson Kempe 2019). The tasks used in the assessment are framed as games and context problems that allow the children to express their ways of experiencing numbers and how they experience *structure* as a means to solve the tasks.

Implementation of a structural approach in teaching

The idea of a structural approach can be found in both larger and smaller intervention studies, but how this is implemented (choice of content, representations and principles for how to teach number as to be relational) differs. We thereby choose to implement two programs that have proven successful in earlier research. Choosing two frameworks that both aim to emphasize structure as the core feature of arithmetic strategies will furthermore give us the opportunity to identify, contrast and conceptualize the facilitating features of the structural approach. That is, not only whether a structural approach is beneficial for learning, but *why* it works, which will be of great importance for developing teaching practices and teacher education.

A. PASMAP (Pattern And Structure Mathematics Awareness Program)

The program, developed by Mulligan and colleagues in Australia (see Mulligan, English & Oslington 2020; Mulligan & Mitchelmore 2009), advocates that pupils' ability to see structure develops through experiencing structure in pattern and number. The program consists of the components; subitizing, unitizing, partitioning, spatial structuring, multiplicative and proportional relationships, and transformation. Experiences offered to pupils then build on pupils' cognitive tendency to seek and analyze patterns. One main idea is that pupils need to be able to analyze and create composed units, e.g. the repeated unit in a pattern (ABCABC), or units in numbers (2+2+2=6). One of several artefacts used in the program is the ten frame (5x2) to model structure, and support counting patterns, grouping, additive facts and inverse relationships.

B. NUSAV (NUmber Structure And Variation program)

The program, developed by Björklund and colleagues in Sweden (see Björklund et al 2018; Kullberg et al. 2020), advocates that pupils' ability to see number structure develops through experiences of part-whole relations of number. The program builds on previous research (Brissiaud 1992; Neuman 1987) emphasizing number relations as the outset for solving arithmetic tasks, which has been further developed in studies with 5- and 7-year-olds, theoretically grounded in variation theory (Marton 2015). The program consists of identified 'critical aspects' of number that pupils need to discern; modes of representation of number, cardinality, ordinality and differentiation of parts and wholes in a number relation (incl. inverse relation between addition and subtraction, and decomposition of number). Finger patterns is used as the main artefact to help pupils see structure in number and number relations.

The two programs are implemented as parallel programs during 8 months. Activities and tasks developed in earlier research are re-designed and implemented through iterations of reflection on the enacted teaching. The teaching is object for analysis and development related to how it reflects the theoretical principles of respective program. Collective meetings are held once a month with teacher group A and B respectively. Video-documentations from their enacting the structural approach in their teaching are used as departure point in the meetings to further the implementation. This process and principles of teacher-researcher collaboration has been used successfully in earlier research (FASETT, DUTTA, EXTENT) and is based in theoretically driven cooperative approaches to professional development (see Sensevy et al. 2018).

To summarize, in our project we will be using the observation protocol to focus on the extent to which a structural approach is implemented, which also will work as our sample selection tool for the intervention. We investigate the children's way of experiencing structure in number concepts and arithmetic, and our intervention programs are theoretically driving the structural approach in teaching number concepts and arithmetic strategies. The content we focus on consistently is *structure* and by using variation theory we are able to analyze and describe *what is afforded in the teaching* (how the idea of structure is coming through) and how this is *reflected in the learning outcome* (how children learn to attend to number structure).

Time plan and realisation

Year	General activities	Part I: Observation of teaching	Part II: Intervention
2021 Spring	Ethical clearance application	Adapting and piloting the observation protocol to Swedish context. Contact with schools.	
2021 Fall	Information to participating schools and teachers	Observations of teaching	Developing (pupil) assessment instrument. Preliminary designs of teaching activities
2022 Spring	Presentation at PME46 conference (focusing teaching practices)	Analysis of teaching observations	First contact with potential participating teachers for teaching intervention. Preparations for (pupil) assessment incl. piloting.
2022 Fall	Presenting at EARLI SIG9 (the theoretical framework). Information to participants/parents		Assessment of pupils. Implementing intervention program A and B.
2023 Spring		Post-Observations of teaching in intervention and comparison groups	Continue implementing intervention program A and B. Post-Assessment of pupils.
2023 Fall	Analysing learning outcomes. Analysing the teaching. Synthesis of teaching and learning outcomes.		
2024 Spring	Reporting results. Presentation at CERME 13 (general findings)		Delayed assessment of pupils' number knowledge
2024 Fall	Final analyses and reporting		

Project organization

The project is a joint commitment where *all researchers are included in designing instruments and teaching activities, conducting the empirical studies, analyzing data and reporting results*.

However, based on each researcher's explicit expertise and experience we take responsibility for the different parts of the project as follows:

Camilla Björklund, project leader works 35% in the project. She is responsible for administration of ethical clearance, and economy (with assisting economy staff). Björklund is responsible for the design, development and administration of the teaching observation protocol and invest substantial time in the analyses and reporting of all parts of the project. Additional time (included in regular employment time) will be added for supervision of PhD student. Björklund is during 2021 engaged in two ongoing projects (activity rate 40%).

Angelika Kullberg, co-researcher, works 30% and will be responsible for developing the pupil assessment instrument and teaching activities. Her expertise in the theoretical frameworks are important for analyzing data, in which she will engage with substantial activity rate.

Anna-Lena Ekdahl, co-researcher, works 35% in the project. Expertise in implementing structural approaches in teaching practice makes her main role to implement the teaching programs with teachers in-practice, including contact with teachers and supporting their implementing the activities and pupil assessments.

Doctoral student, works 30% in the project as part of the PhD-studies. Assisting in collecting data from teaching observations, pupil assessments and implementing the teaching activities.

Equipment and need for research infrastructure

The host department holds a videolab (The former LinCS), a national centre of excellence funded by the Swedish Research Council, with excellent infrastructure for storage, archive, sharing, analyzing and utilizing video-data from social sciences research. The technical equipment and knowledge for analysis of complex data that is available at LinCS, does not require any additional costs.

International and national collaboration

The research group constituting SATSA has an extensive international network of researchers within the field of mathematics education. Björklund is member of working groups and interests groups (CERME and POEM) concerning early mathematics with re-occurring activities that Björklund is actively initiating (special issues in journals, symposia, conferences etc.). Kullberg is an active member in PME (Psychology of Mathematics Education) and council member of World Association of Lesson Studies including a network of researchers working with practice-based educational research. Ekdahl is a visiting associate at University of the Witwatersrand and has a close collaboration with the *WITS Maths Connect Primary* research group.

The SATSA project is heavily based on theoretical principles that are to be implemented into teaching, and analyses to be done on many levels. Therefore, we have contact with three experts in the theoretical frameworks we are addressing: Professor *Ulla Runesson Kempe* (Jönköping University) has in several earlier projects and colloquium collaborated with the research group and has been a front figure in the development of practice-based research using variation theory as theoretical foundation. Professor *Hamsa Venkat* (University of the Witwatersrand, South Africa) is also a collaborator and advisor in earlier projects with extensive expertise in the structural approach and instruments for assessing teaching quality. Additionally, we have ongoing discussions with professor *Lyn English* (Queensland University of Technology, QUT Australia) one of the researchers in PASMAP, who has shown an explicit interest in our research and welcome further common research activities.

Other applications or grants

The main applicant Björklund is applying for a network grant EMELI (Early Mathematics Education studies of Longitudinal Interventions) from the Swedish Research Council in 2020, to establish cooperation between international research groups working with early mathematics education research. The current project SATSA has different aim and research questions but will benefit from the network's activities through common interests in practice-based research to enhance teaching on theoretical and empirical foundations.

References

Askew, M., Venkat, H., et al. (2019). Teaching for structure and generality: Assessing changes in teachers mediating primary mathematics. In M. Graven, H. Venkat, A. A. Essien & P. Vale (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education.* (pp. 41-48). Pretoria, South Africa: PME.

Baroody, A. J. (1987). *Children's mathematical thinking*. New York: Teachers College Press.

Baroody, A., & Purpura, D. (2017). Early number and operations: Whole numbers. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 308–354). Reston, VA: National Council of Teachers of Mathematics.

Björklund, C. (in review). Children's awareness of numbers' part-whole relations when bridging through 10.

Björklund, C., Alkhede, M., et al. (2018). Teaching finger patterns for arithmetic development to preschoolers. In J. Häggström, Y. Liljekvist, J. Bergman Ärlebäck, M. Fahlgren & O. Olande (Eds.), *Proceedings of MADIF 11 The eleventh research seminar of the Swedish Society for Research in Mathematics Education*. Karlstad, January 23–24, 2018.

Björklund, C., & Ekdahl, A-L. (in review). Learning to teach mathematics in preschool through theory-driven interventions.

Björklund, C., Ekdahl, A-L., & Runesson Kempe, U. (in review). Implementing a Structural Approach in Preschool Number Activities. Effects of an Intervention Program.

Björklund, C., Marton, F., & Kullberg, A. (manuscript). What is to be learned? Critical aspects of elementary arithmetic skills.

Björklund, C., & Runesson Kempe, U. (2019). Framework for analysing children's ways of experiencing numbers. *Proceedings from CERME11, 2019*, Utrecht, the Netherlands.

Brissiaud, R. (1992). A tool for number construction: Finger symbol sets. In J. Bideaud, C. Meljac, & J.-P. Fischer (Eds.), *Pathways to number: Children's developing numerical abilities* (pp. 41-65). Hillsdale, NJ: Lawrence Erlbaum.

Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., & Hiebert, J. (2017). Making classroom implementation an integral part of research. *Journal for Research in Mathematics Education, 48*(4), 342–347.

Carraher, D.W., Schliemann, A.D., Brizuela, B.M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *JRME*, *37*(2), 87-115.

Cheng, Z-J. (2012). Teaching young children decomposition strategies to solve addition problems: An experimental study. *The Journal of Mathematical Behavior, 31*(1), 29–47.

Clements, D., & Sarama, J. (2011). Early Childhood Mathematics Intervention. *Science, 333* (6045), 968–970. DOI: 10.1126/science.1204537

Davydov, V. V. (1982). The psychological characteristics of the formation of elementary mathematical operations in children. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, NY: Lawrence Erl.

Duncan, G. J., Dowsett, C. J., et al. (2007). School readiness and later achievement. *Developmental Psychology, 43*(6), 1428–1446.

Ekdahl, A-L. (accepted). Different Learning Possibilities from the same Activity - Swedish Preschool Teachers' Enactments of a Number Relation Activity. *Scandinavian Journal of Educational Research.*

Ekdahl, A-L. (2019). *Teaching for the learning of additive part-whole relations: The power of variation and connections*. (Diss.). Jönköping University.

Ekdahl, A-L., Venkat, H., & Runesson, U. (2016). Coding teaching for simultaneity and connections. *Educational Studies in Mathematics, 93*(3), 293–313.

Ellemor-Collins, D., & Wright, R. B. (2009). Structuring numbers 1 to 20: Developing facile addition and subtraction. *Mathematics Education Research Journal, 21*(2), 50–75.

Fuson, K. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. Macmillan Library Ref.

Gage, N. L. (1978). *The scientific basis of the art of teaching.* New York: Teachers College.

Gersten, R., Jordan, N. C., & Flojo, J. R. (2005). Early identification and interventions for students with mathematical difficulties. *Journal of Learning Disabilities, 38*, 293–304.

Gibson, J. J., & Gibson, E. J. (1955). Perceptual learning: Differentiation or enrichment. *Psychological Review, 62*, 32–41.
Hiebert, J. & Grouws, D. (2007). The effects of classroom mathematics teaching on student's learning. *Second handbook of research on mathematics teaching and learning.* (pp. 371–404). Charlotte, NC: Information Age Publishing.
Kullberg, A., Björklund, C., Brkovic, I., & Runesson Kempe, U. (2020). Effects of learning addition and subtraction in preschool by making the first ten numbers and their relations visible with finger patterns. *Educational Studies in Mathematics, 103*, 157-172.
Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge.
Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah N.J.: Lawrence Erlbaum.
Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10–32.
Mulligan, J., English, L., & Oslington, G. (2020). Supporting early mathematical development through a 'pattern and structure' intervention program. *ZDM Mathematics Education*.
Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33-49.
Mulligan, J. T., Mitchelmore, M. C., English, L. D., & Crevensten, N. (2013). Reconceptualizing early mathematics learning: The fundamental role of pattern and structure. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 47–66). Dordrecht, the Netherlands: Springer.
Mulligan, J. T. & Vergnaud, G. (2006). Research on children's early mathematical development: Towards Integrated perspectives. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education*. London: Sense.
Neuman, D. (1987). *The origin of arithmetic skills: A phenomenographic approach*. Göteborg: Acta Universitatis Gothoburgensis.
Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237-269.
Polotskaia, E., & Savard, A. (2018). Using the Relational Paradigm: Effects on pupils' reasoning in solving additive word problems*. Research in Mathematics Education, 20*(1), 70-90.
Schmittau, J. (2003). Cultural-historical theory and mathematics education. In A. Kozulin, B. Gindis, V. S. Ageyev, & S. M. Miller (Eds.), *Vygotsky's educational theory in cultural context* (pp. 225–245). Cambridge: Cambridge University Press.
Schollar, E. (2015). *The Primary Mathematics Research Project: 2004-2012.* (Diss). University of Cape Town: Department of Sociology.
Sensevy, G., Quilio, S., et al. (2018). How teachers and researchers can cooperate to (re)design a curriculum. In Yoshinori Shimizu & Renuka Vithal (Eds.), *ICMI Study 24 Conference Proceedings*. Tsukuba, Japan. University of Tsukuba.
Sterner, G., Wolff, U., & Helenius, O. (2019.) Reasoning about representations: Effects of an early math intervention. *Scandinavian Journal of Educational Research*, online first.
Swedish Government (2020). *En tioårig grundskola*. Dir. 2020:24.
Wang, A., Firmender, J., Power, J., & Byrnes, J. (2016). Understanding the program effectiveness of early mathematics interventions for prekindergarten and kindergarten environments: A meta-analytic review. *Early Education and Development, 27*(5), 692–713.
Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal*, *15*(2), 122-137.
Venkat, H., Askew, M., Watson, A., & Mason, J. (2019). Architecture of Mathematical Structure. *For the Learning of Mathematics*, *39*(1), 13-17.
Venkat, H. & Askew, M. (2018). Mediating primary mathematics: Theory, concepts, and a framework for studying practice. *Educational Studies in Mathematics, 97,* 71–92.
Venkat, H., Askew, M., Morrison, S. & Abdulhamid, L. (2019). *Shifts in Early Number learning in South Africa*. Paper at the PME 43, July 7-12, 2019, Pretoria, South Africa.
Zhou, Z., & Peverly, S. T. (2005). Teaching addition and subtraction to first graders: A Chinese perspective. *Psychology in the Schools, 42*(3), 259–272.